DETERMINING THE PARAMETERS OF SUBSONIC FLOW IN A CHANNEL BEYOND THE ZONE OF AXIAL INHOMOGENEITY OF WEAK DISTURBING FORCES AND HEAT SOURCES

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In gas dynamics one is frequently obliged to consider flows of a medium in a channel in a field of disturbing forces **F** and heat sources Q. In those cases where the disturbing factors are relatively small, the equations can be linearized near the solution for $\mathbf{F} = 0$, $Q = 0$. In many practical applications **F** and Q are inhomogeneous in the longitudinal and transverse directions over some segment of finite length(*) (which we shall denote by L), while upstream from this segment $\mathbf{F} = 0$, $Q = 0$ and downstream from it **F** and Q depend practically only on the coordinates in the transverse cross section (or are equal to zero in a particular case).

We shall show that in the linear formulation for subsonic flows in a flat channel and circular pipe it is possible to find the flow parameters beyond the zone L (in the segment L' separated from L by some distance l, Fig. 1) without solving the corresponding linear system of partial differential equations. The flow in the segment L' is determined by **F** and Q in L' (which depend only on the transverse coordinates) and by the integrals of these quantities over the segment L.

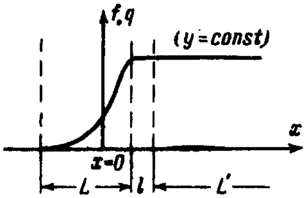

Fig. 1

In dimensionless variables the equations of gas dynamics for a perfect gas with constant heat capacities are

$$\rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + N\mathbf{f}, \qquad \operatorname{div}\rho\mathbf{v} = 0$$

$$\rho\mathbf{v}\nabla\varepsilon = -p\operatorname{div}\mathbf{v} + Nq, \qquad p = (\gamma - 1)\rho\varepsilon \qquad (1)$$

Here the density ρ, velocity **v**, pressure p, internal energy ε, and Cartesian coordinates x, y, z are referred to $\rho_*, V, \rho_* V^2, V^2$, and h, respectively (ρ_* is the characteristic density V the average velocity over the channel cross section, and h the characteristic transverse dimension); γ is the ratio of specific heats; $N\mathbf{f}$ and Nq are the dimensionless densities of forces and heat sources; N is a parameter characterizing the relative magnitude of the disturbing factors.

Henceforth we assume that **f** and q depend explicitly on the flow parameters (but not on their derivatives) and on the coordinates.

If $N \ll 1$, the solution of (1) can be sought in the form of series,

$$\rho = \rho_0 + N\rho_1 + \ldots, \qquad \varepsilon = \varepsilon_0 + N\varepsilon_1 + \ldots$$

$$p = p_0 + Np_1 + \ldots, \qquad \mathbf{v} = \mathbf{v}_0 + N\mathbf{v}_1 + \ldots \qquad (2)$$

where the quantities with the subscript 0 satisfy system (1) for $N = 0$. For flow in a channel with a constant cross section (along the axial coordinate x) and gas-impermeable walls we have

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*) In magnetohydrodynamics **F** and Q are the electromagnetic force and the Joule dissipation, respectively. In many practically interesting cases the length of the zone L is comparable to the height of the channel.

$$\mathbf{v}_0 = (u_0(y, z), 0, 0), \qquad \rho_0 = \rho_0(y, z), \qquad p_0 = p_{00} = \text{const} \tag{3}$$

Here u_0 and ρ_0 are arbitrary smooth nonvanishing functions(*).

The equations of the first approximation for a flat channel $|x| < \infty$, $0 < y < 1$ and circular pipe $y < 1$, $0 < \theta < 2\pi$, $|x| < \infty$ are of the form(**)

$$\rho_0 u_0 \frac{\partial u_1}{\partial x} + \rho_0 v_1 \frac{du_0}{dy} + \frac{\partial p_1}{\partial x} = f_x, \qquad \rho_0 u_0 \frac{\partial v_1}{\partial x} + \frac{\partial p_1}{\partial y} = f_y$$

$$u_0 \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial u_1}{\partial x} + v_1 \frac{d\rho_0}{dy} + \frac{\rho_0}{y^\nu} \frac{\partial}{\partial y}(y^\nu v_1) = 0$$

$$u_0 \frac{\partial p_1}{\partial x} - a_0^2 u_0 \frac{\partial \rho_1}{\partial x} - a_0^2 v_1 \frac{d\rho_0}{dy} = q^\circ \qquad (q^\circ = (\gamma - 1) q), \qquad \rho_0 u_0 \frac{\partial w_1}{\partial x} = f_z \tag{4}$$

In System (4) the quantities u_1, v_1, and w_1 are the projections of the perturbed velocity vector on the axes x, y, and z (or θ). respectively; $\nu = 0$ for a flat channel and $\nu = 1$ for a circular pipe; the functions ρ_0 and u_0 depend only on y; a_0 is the speed of sound computed from parameters (3). The disturbing factors $\mathbf{f}$ and q depend on the coordinates x, y and on gas dynamic parameters (3); they are assumed to be known(***). The last Eq. in (4) does not depend on the other equations and determines the twist of the stream.

We assume that $\mathbf{f}$ and q are inhomogeneous with respect to x and y in the segment L; to the left of L (for $x \to -\infty$) $\mathbf{F} = 0$, $q = 0$, while to the right of it $\mathbf{f}$ and q depend only on y. Mathematically this assumption can be expressed by requiring convergence of the integrals

$$\int_{-\infty}^{0} \eta\, dx, \quad b(y) = \int_{0}^{\infty} (\eta - \eta_\infty)\, dx$$

and fulfillment of the approximate Eq.

$$\int_{0}^{x} (\eta - \eta_\infty)\, dx = b$$

for x lying to the right of L. (Here $\eta(x, y)$ is either of the functions $\mathbf{f}$, q; $\eta_\infty = \eta(\infty, y)$; the cross section $x = 0$ belongs to the zone L).

Let us consider the boundary conditions for system (4). We shall examine subsonic flows (3). These will be unperturbed (by the factors $\mathbf{f}$ and q) if the conditions at the channel exit upon "actuation" of the disturbing factors are adjusted in such a way that the conditions at the entrance are not altered. In the general case by u_0, ρ_0, and p_{00} we mean certain distributions of the gas dynamic parameters in the entrance cross section which come into being upon the actuation of the disturbances $\mathbf{f}$ and q.

In these cases the perturbations of v_1, ρ_1, and p_1 are equal to zero as $x \to -\infty$. At the impermeable walls of the channel (for $y = 1$) we have $v_1 = 0$.

Let us analyze system (4). Integrating the first, third, and fourth of its Eqs. over x in the range $(-\infty, x)$, we obtain three relations which enable us to express u_1, p_1, and ρ_1 in terms of the integrals of v_1 and $\partial v_1/\partial y$. Then, differentiating the first and second Eqs. of system (4) over y and x, respectively, and taking account of the resulting expression for u_1, we obtain one partial differential equation for the velocity v_1,

$$\rho_0 u_0 \left(\alpha \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial x^2}\right) + \frac{\partial v_1}{\partial y} \frac{u_0}{y^\nu} \frac{d}{dy}(\alpha \rho_0 y^\nu) - v_1 \frac{d}{dy}\left[\alpha \rho_0 y^\nu \frac{d}{dy}\left(\frac{u_0}{y^\nu}\right)\right] = \Phi$$

$$\alpha = (M_0^2 - 1)^{-1}, \quad \Phi = \frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x} - \frac{\partial}{\partial y}(s u_0 \rho_0), \qquad s = \frac{\alpha}{\rho_0 a_0^2}(u_0 f_x - q^\circ) \tag{5}$$

$$v_1 = 0 \quad \text{при } x \to -\infty, \qquad v_1 = 0 \quad \text{при } y = 0,\ y = 1$$

*) The presence of discontinuities in the derivatives of u_0, ρ_0 would result in a discontinuous profile of the perturbed velocity u_1; the presence of points where $\rho_0 u_0 = 0$ would result in an unlimited increase in u_1 (Formula (14)). The limitations imposed on the functions u_0 and ρ_0 are dictated by the model of a nonviscous medium which we are using.

**) In the case of a circular pipe y, θ, x are cylindrical coordinates.

***) The fact that f and q depend only on x and y allowed us to assume that $\partial/\partial z = 0$ (or $\partial/\partial\theta = 0$) in deriving (4).

Here M_0 is the Mach number for unperturbed flow (3). For $M_0 < 1$ Eq. (5) has an elliptic character. By hypothesis, the disturbing factors are axially homogeneous to the right of L and

$$\Phi = \Phi_\infty(y) = \frac{df_{x\infty}}{dy} - \frac{d}{dy}(su_0\rho_0) = \frac{dS}{dy}, \qquad S = -\alpha u_0\left(\frac{f_{x\infty}}{u_0} - \frac{q^\circ_\infty}{a_0^2}\right) \tag{6}$$

For an incompressible fluid we must set $\rho_0 \equiv 1$, $\alpha = -1$, $a_0 = \infty$ in (5) and (6).

Due to the elliptic character of Eq. (5), the perturbations occasioned by the axial inhomogeneity of Φ on L are damped out to the right of L over the length Δx of magnitude on the order of $l = (1 - M_*^2)^{1/2}$, where M_* is the Mach number of unperturbed flow averaged over the cross section(*) (the dimensional damping length can be obtained by multiplying Δx by h). Hence, at a sufficient large distance Δx from the right end of L ($\Delta x \gtrsim l$) the velocity v_1 practically ceases to depend on x ($v_1 = v_1^+(y)$) and is given by Eq. (**)

$$\alpha\rho_0 u_0 \frac{d^2 v_1^+}{dy^2} + \frac{dv_1^+}{dy}\frac{u_0}{y^\nu}\frac{d}{dy}(\alpha\rho_0 y^\nu) - v_1^+ \frac{d}{dy}\left(\alpha\rho_0 y^\nu \frac{d}{dy}\frac{u_0}{y^\nu}\right) = \frac{dS}{dy}$$

$$v_1^+(0) = 0, \qquad v_1^+(1) = 0 \tag{7}$$

The solution of this Eq. is of the form

$$v_1^+ = \frac{u_0}{y^\nu}\int_0^y (S + C)\,\mu\, dy, \qquad C = -\int_0^1 S\mu\, dy \Big/ \int_0^1 \mu\, dy \cdot \qquad \left(\mu = \frac{y^\nu}{\alpha\rho_0 u_0^2}\right) \tag{8}$$

According to this expression, $v_1^+ \equiv 0$ for $S = \text{const}$. This is possible, for example, for $f_{x\infty} = 0$, $q_\infty^\circ = 0$, or $f_{x\infty} = \text{const}$, $q^\circ_\infty = \text{const}$ and a homogeneous unperturbed stream. In the case of an incompressible fluid $S = f_{x\infty}$ and the velocity v_1^+ can differ from zero only with an inhomogeneous distribution of $f_{x\infty}$.

Let us determine the remaining flow parameters to the right of the zone L, i.e. in the zone L', where $v_1 \approx v^+_1$. The flow in this zone will henceforth be called pseudodeveloped (***); its parameters will be accompanied by the superscript +.

Let us integrate the first, third, and fourth Eqs. of system (4) over x within the limits $(-\infty, x)$, where x belongs to the pseudodeveloped flow zone. We obtain

$$\rho_0 u_0 u_1^+ + p_1^+ + \frac{du_0}{dy}(\psi + x\rho_0 v_1^+) = x f_{x\infty} + \xi_1$$

$$u_0\rho_1^+ + \rho_0 u_1^+ + \frac{d\psi}{dy} + \frac{\nu}{y}\psi + x\left[\frac{d}{dy}\rho_0 v_1^+ + \frac{\nu}{y}\rho_0 v_1^+\right] = 0 \tag{9}$$

$$u_0 p_1^+ - a_0^2 u_0 \rho_1^+ - \frac{a_0^2}{\rho_0}\frac{d\rho_0}{dy}(\psi + x\rho_0 v_1^+) = q_\infty^\circ x + \xi_2$$

$$\psi = \psi(y) = \int_{-\infty}^0 \rho_0 v_1\, dx + \int_0^\infty \rho_0 (v_1 - v_1^+)\, dx \tag{10}$$

$$\xi_1 = \xi_1(y) = \int_{-\infty}^0 f_x\, dx + \int_0^\infty (f_x - f_{x\infty})\, dx, \quad \xi_2 = \xi_2(y) = \int_{-\infty}^0 q^\circ\, dx + \int_0^\infty (q^\circ - q_\infty^\circ)\, dx$$

Instead of the superscript ∞ the integrals of (10) should, strictly speaking, contain the quantity x. But since x belongs to the pseudodeveloped flow zone, where v_1, f_x, and q° practically coincide with v_1^+, $f_{x\infty}$ and q_∞° (i.e. in theory they reach these asymptotic values

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*) A very rough and in most cases exaggerated estimate is used. An exact estimate can, of course, be obtained after solving Eq. (5), We also note that numbers M_* close to unity are excluded from consideration. As $M_* \to 1$ the perturbation of the velocity u_1 increases without limit and the linear theory no longer applies.

**) Theoretically $v_1 \to v_1^+(y)$ as $x \to \infty$. However, the asymptotic form is determined by the exponential factor, and transition to the profile $v_1^+(y)$ occurs at a finite distance from L equal to l in order of magnitude.

***) This term is used in monograph [1] to describe the flow of a nonviscous incompressible fluid in a flat channel beyond the inhomogeneous magnetic field zone.

very quickly), replacement of the upper limits is quite permissible, and ξ_1, ξ_2, and ψ can be considered as functions of the single variable y. The quantities ξ_1 and ξ_2 are assumed known.

From the second Eq. of system (4) we have

$$p_1^+ = \xi_3(y) + \varepsilon(x) \qquad \left(\xi_3 = \int_0^y f_{y\infty}\, dy\right) \tag{11}$$

Here ξ_3 is known and $\varepsilon(x)$ must be determined. Substituting (8) and (11) into relations (9) and eliminating u_1^+ and ρ_1^+, we obtain

$$\frac{d\psi}{dy} - \psi\frac{d}{dy}\ln\frac{u_0\rho_0}{y^\nu} = \Gamma, \qquad \Gamma = t(y) - \frac{k}{\alpha u_0}, \qquad t = \frac{\xi_2}{a_0^2} - \frac{\xi_1}{u_0} - \frac{\xi_3}{u_0\alpha}$$

$$k = Cx + \varepsilon(x), \qquad \psi(0) = 0, \qquad \psi(1) = 0 \tag{12}$$

Here the constant C is given by Formula (8). All of the quantities in (12) except k depend only on y. The quantity k depends only on x. Hence, k = const. The existence of two boundary conditions for ψ makes it possible to determine $\psi(y)$ and k. Solution (12) can be written as

$$\psi = \frac{u_0\rho_0}{y^\nu}\int_0^y \frac{\Gamma y^\nu}{\rho_0 u_0}\, dy, \qquad k = \int_0^1 \frac{t y^\nu\, dy}{\rho_0 u_0} \Bigg/ \int_0^1 \frac{y^\nu\, dy}{\alpha\rho_0 u_0^2} \tag{13}$$

Using (9) and (13), we can find all of the pseudodeveloped flow parameters,

$$u_1^+ = \frac{1}{\rho_0 u_0}\left\{x\left(f_{x\infty} + C - \rho_0 v_1^+ \frac{du_0}{dy}\right) - \xi_3 - k + \xi_1 - \psi\frac{du_0}{dy}\right\}$$

$$\rho_1^+ = -\frac{1}{u_0}\left\{\rho_0 u_1^+ + \frac{1}{y^\nu}\frac{d}{dy}(y^\nu\psi + \rho_0 v_1^+ y^\nu x)\right\}, \qquad p_1^+ = \xi_3 + k - Cx \tag{14}$$

In accordance with (4), the transverse velocity w_1^+ is given by Formula

$$w_1^+ = \frac{1}{\rho_0 u_0}\xi_4 + x\frac{f_{z\infty}}{\rho_0 u_0} \qquad \left(\xi_4 = \int_{-\infty}^0 f_z\, dx + \int_0^\infty (f_z - f_{z\infty})\, dx\right) \tag{15}$$

For an incompressible fluid for $\mathbf{f}$ = const we find from (13) and (14) ($\rho_0 \equiv 1$, $\alpha = -1$, $a_0^2 = \infty$, $C = -f_{x\infty}$, $v_1^+ \equiv 0$) that

$$u_1^+ = \frac{1}{u_0}\left(\xi_1 - k - \psi\frac{du_0}{dy} - y f_{y\infty}\right), \qquad p_1^+ = f_{x\infty}x + f_{y\infty}y + k$$

$$\psi = \frac{u_0}{y^\nu}\int_0^y \Gamma\frac{y^\nu\, dy}{u_0}, \quad k = -\int_0^1 \frac{t y^\nu\, dy}{u_0} \Bigg/ \int_0^1 \frac{y^\nu\, dy}{u_0^2}, \quad \Gamma = t + \frac{k}{u_0}, \quad t = \frac{y f_{y\infty}}{u_0} - \frac{\xi_1}{u_0} \tag{16}$$

The characteristics of pseudodeveloped flows have been determined (for the simplest cases) in the field of magnetohydrodynamics. Thus, assuming that $u_0 \equiv 1$, Shercliff [2] found the asymptotic velocity profile for the flow of an isotropically conducting fluid in a magnetohydrodynamic flowmeter. The corresponding result can be obtained from (16) by substituting in $\nu = 0$, $\mathbf{f}_\infty = 0$, $u_0 \equiv 1$, and determining ξ_1 from the solution of the problem of electric field distribution in a channel with nonconductive walls [2 and 3]. The results obtained above are extended for the case of an anisotropically conducting fluid and an inhomogeneous unperturbed flow in [4].

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